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1978 J. Phys. A: Math. Gen. 11 L57

(http://iopscience.iop.org/0305-4470/11/3/004)

J. Phys. A: Math. Gen., Vol. 11, No. 3, 1978. Printed in Great Britain

LETTER TO THE EDITOR

Monte Carlo studies of two-dimensional percolation

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Received 12 January 1978

Abstract. A one-spin flip Ising model is used to provide data on cluster statistics for random and Ising percolation. The concentration p is controlled by the magnetic field. At sufficiently high temperatures the system corresponds to random percolation, and the theoretical formula $s/n = (1-p)/p$ is verified for large clusters at critical concentration p_c (s = number of boundary sites). It is also found that the relation is accurately satisfied for all percolating clusters when $p > p_c$ but not for Ising percolation at temperature $2T_c$. For random percolation with $p > p_c$ the finite n-clusters are found to follow an asymptotic decay of the form $\exp(-b(p)n^{1/2})$ in accord with theory.

The two-dimensional one-spin flip Ising model (Glauber model) has been used previously to provide statistical data on Ising clusters in zero magnetic field as a function of temperature (Domb *et al* 1975, Domb and Stoll 1977). By keeping the temperature constant, and varying the magnetic field to change the concentration, similar data can be derived from this model for percolation. If the temperature is sufficiently high for the coupling between spins to be negligible the data correspond to random percolation. For lower temperatures the data correspond to correlated or Ising percolation which has begun to attract attention recently (Müller-Krumbhaar 1974, Coniglio 1975). A similar approach to the above has been used recently by Odagaki *et al* (1975).

Details of the Monte Carlo procedure and its limitations in the simulation of an infinite system are given elsewhere (Stoll *et al* 1973, Schneider and Stoll 1975). The system considered is a 110×110 simple quadratic lattice subject to periodic boundary conditions. An important aspect is the averaging of the data over a large number of configurations to obtain data with sufficiently small sampling errors. We have adopted the following procedure.

A total number of 40 000 configurations is generated for each concentration p. As a first step 200 successive configurations are grouped together and averaged linearly; an averaging and smoothing procedure is then used on the 200 groups.

Let us denote by $P_{ij}(n)$ the number of n-clusters in the jth configuration of the ith group; this is a function of p but for convenience we will not show this dependence explicitly. We first form

$$P_i(n) = \frac{1}{200} \sum_{j=1}^{200} P_{ij}(n), \tag{1}$$

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0305-4770/78/0003-9057 $01.00

and the standard deviation is given by the usual formula

$$\Delta P_i(n)=\left(\sum_{j=1}^{200}(P_{ij}(n)-P_i(n))^2/199\right)^{1/2}. \tag{2}$$

In the next step we average for fixed n the 200 $P_i(n)$. In this way we obtain the mean value $P(n)$ and the standard deviation $\Delta_i P(n)$ defined by

$$P(n)=\frac{1}{200}\sum_{i=1}^{200}P_i(n) \tag{3}$$

and

$$\Delta_i P(n)=\left(\sum_{j=1}^{200}(P_j(n)-P(n))^2/199\right)^{1/2}. \tag{4}$$

In the case where all $P_{ij}(n)$ are independent $\Delta_i P(n)$ and $\Delta P_i(n)$ are related by

$$\frac{1}{200}\sum_{i=1}^{200}(\Delta P_i(n))^2\approx\langle(\Delta P(n))^2\rangle=200(\Delta_i P(n))^2. \tag{5}$$

If the P_{ij} are correlated relation (5) is no longer valid. In this case the right-hand and left-hand side expressions in (5) are related by a number K,

$$K=200\frac{(\Delta_i P(n))^2}{\langle(\Delta P(n))^2\rangle}>1. \tag{6}$$

K is a measure of the correlation. We expect the uncertainty of $P(n)$, namely $\delta P(n)$, to increase by a factor of $\sqrt{K}$ so that in analogy to the statistics of an independent distribution:

$$\delta P(n)=[\langle(\Delta P(n))^2\rangle K/40000]^{1/2}. \tag{7}$$

Furthermore for large n the fluctuations remain of the order $P(n)$, and the function $P(n)$ is therefore 'folded' by a triangular function in a manner analogous to the use of high-frequency filters in electrical engineering. The width of the triangle is adjusted so that the $P(n)$ with good statistics are smoothed over a small range of n, whilst those with poor statistics are averaged over a large range.

Define a triangle function $T(j, x)$ with the following properties:

$$T(j,x)=\begin{cases}0 & |j|>\sigma x\\ A\left(1-\dfrac{j}{\sigma x}\right) & 0\leqslant j\leqslant\sigma x\\ A\left(1+\dfrac{j}{\sigma x}\right) & -\sigma x\leqslant j\leqslant 0.\end{cases} \tag{8}$$

The constant A is chosen so that

$$\sum_j T(j,x)=1. \tag{9}$$

The smoothed values $\bar{P}(n)$ are then given by

$$\bar{P}(n)=\sum_j P(n-j)T(j,\delta P(n-j)/P(n-j)). \tag{10}$$

The value of σ used in our calculations was 100.

Denoting the number of perimeter sites in a cluster by s, the parameter $s/n(=a)$ is of considerable importance in percolation (Domb 1976), and at the critical concentration its limiting value for large clusters has been shown to be (Leath 1976)

$$a_c = \frac{1-p_c}{p_c}. \tag{11}$$

We first used the data to test relation (11) and to investigate the behaviour of s/n for large clusters as a function of p. The results are shown in figure 1. Relation (11) is seen to be well satisfied, but in addition we found that for the percolating cluster ($p > p_c$) a relation analogous to (11) is well satisfied for all p,

$$\left(\frac{s}{n}\right)_{\text{percolating}} = \frac{1-p}{p}. \tag{12}$$

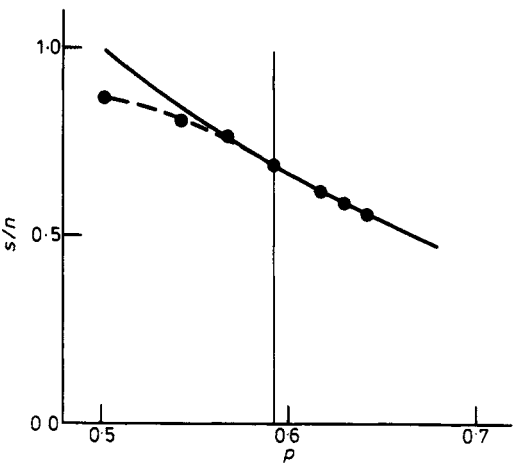

Figure 1. s/n against p for random percolation (limiting values for large n). Points correspond to Monte Carlo data, and the full curve represents $(1-p)/p$.

The accuracy with which relation (12) is satisfied is indicated by the detailed numerical data in table 1. It will be seen that the maximum error arises at $p = p_c$ where it is known that the relation is exact. It is therefore reasonable to conclude that relation (12) is exact for $p > p_c$. We have since learned that Dr Alex Hankey has derived relation (12) independently on theoretical grounds (Hankey 1978).

Table 1. Comparison of s/n with $(1-p)/p$ for percolating clusters.

p	$\frac{1-p}{p}$	$\left(\frac{s}{n}\right)_{\text{percolating}}$	$\left[\left(\frac{s}{n}\right)_{\text{percolating}}\frac{p}{1-p}-1\right]\times 10^3$	$\frac{\delta(s/n)}{s/n}\times 10^3$
0·593 (p_c)	0·68634	0·68010	−9·10	±0·3
0·618	0·61802	0·61812	−0·17	
0·6305	0·58604	0·58601	−0·05	
0·643	0·55521	0·53557	+0·65	

For $p < p_c$ the data fitted a relation of the form

$$\frac{s}{n} \simeq \frac{1-p}{p} - A(p_c - p)^{\theta}. \tag{13}$$

Such a relation has been suggested by Stauffer (1978) on the basis of a scaling hypothesis with θ equal to $\beta\delta - 1$. Whilst our data are consistent with this hypothesis ($A = 60$, $\theta \approx 2$) they are not conclusive since the data can be fitted equally well by other values of (A, θ).

When correlations are present between overturned spins there is no longer any theoretical support for any formulae of type (11) or (12). In figure 2 we have plotted the data for $T = 2T_c$ in the same way as in figure 1, and it will be seen that the formulae are no longer satisfied. One must perhaps seek new relations involving suitable parameters of the two-dimensional Ising model in a non-zero field.

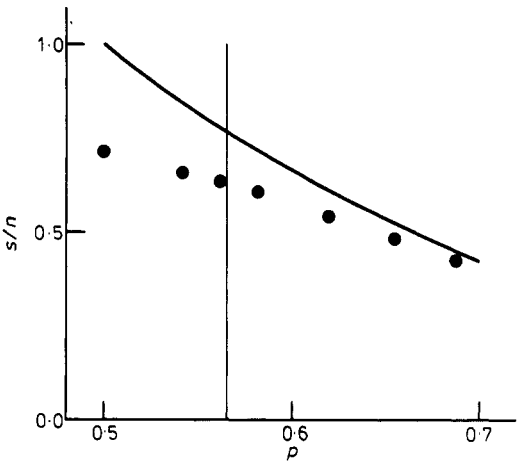

Figure 2. s/n against p for Ising percolation with $T = 2T_c$ (limiting values for large n). Points correspond to Monte Carlo data, and the full curve represents $(1-p)/p$.

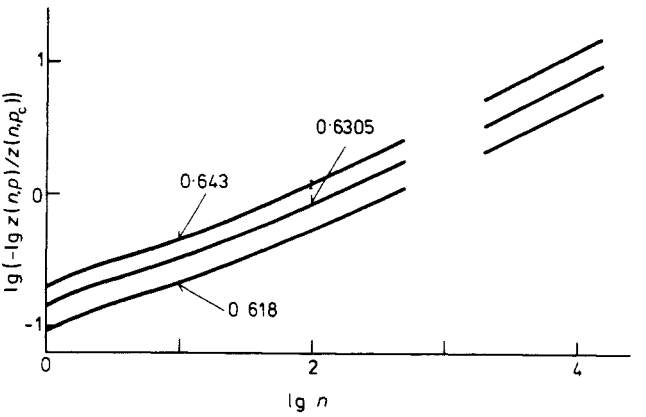

Figure 3. $\lg(-\lg z(n, p)/z(n, p_c))$ against $\lg n$.

Finally we examined the distribution of finite n-clusters for $p > p_c$. For sufficiently large clusters there is theoretical support (Stauffer 1978, Kunz and Souillard 1977, Hankey 1978) for the formula

$$z(n, p) \sim A(n) \exp(-b(p)n^{1/2}) \tag{14}$$

in two dimensions, where $z(n, p)$ is the number of n-clusters at concentration p. To test (14), we plotted the function

$$\lg[-\lg(z(n, p)/z(n, p_c))], \tag{15}$$

against $\lg n$, and the results are shown in figure 3 for $p = 0{\cdot}618$, $0{\cdot}6305$, and $0{\cdot}643$. It will be seen that the curves approach linearity for large n, and our estimates of $0{\cdot}48$–$0{\cdot}49$ of the limiting slopes are quite close to the theoretical value of $\frac{1}{2}$.

We are grateful to T Schneider, D Stauffer and A Hankey for helpful discussion and correspondence.

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